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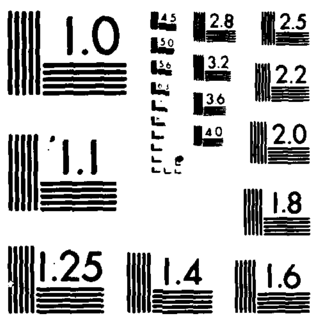

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THE DETERMINATION OF CRITICAL FLUTTER CONDITIONS OF NONLINEAR SYSTEMS

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SUMMARY

A flutter analysis procedure for nonlinear systems is proposed as an alternative to timewise integration methods. It is based on an energy method due to J. Roorda and S. Nemat-Nasser; and shows promise of being a practical procedure provided the number of parameters can be minimised by the representation of the flutter system by an equivalent (condensed) two degree of freedom system in the neighbourhood of the critical condition. The relationship to the simpler method put forward by R.F. Taylor *et alia* is considered.

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1 INTRODUCTION

The theoretical prediction of flutter characteristics can normally be treated adequately as a linear, small perturbation, problem. However there may be circumstances where a nonlinear analysis is essential. This may sometimes be the case, for example, in transonic flow conditions. One obvious method that could be used would be to determine by numerical, timewise, integration the motion consequent upon a number of initial disturbances. So by interpolation one should be able to determine possible states of periodic motion. Alternatively one can attempt to determine directly such limit cycles. This present paper is, therefore, an initial consideration of how this might be done. An approximate method, with two possible simplifications, is suggested.

2 THE BASIC RELATIONSHIPS

Suppose we have an aeroelastic system where Lagrange equation of motion can be approximated sufficiently well, *for the purpose* of finding periodic solutions, by the nonlinear matrix differential equation:-

$$A\nu^2 q'' + \left(B\nu + \frac{D\nu}{v}\right) q' + \left(C + \frac{E}{v^2}\right) q$$
$$+ f(q,\nu q') + \frac{1}{v^2}\, g(q,\nu v q') = 0 \quad . \qquad (2\text{-}1)$$

Here A is the inertia matrix, B and D the linear aerodynamic and structural damping matrices respectively, C and E the linear aerodynamic and structural stiffness matrices respectively, f is a column vector of the nonlinear aerodynamic terms, and g is a column vector of the nonlinear structural terms. A non-dimensional time $\tau\,(= \omega t)$ has been introduced where ω is an, as yet, undetermined frequency. The frequency parameter based on this frequency, the airspeed and a chosen reference length is denoted by ν . The primes denote differentiation with respect to τ . v is an airspeed parameter, being the ratio of the airspeed to some reference speed. The matrices A and E will both be symmetric and positive definite, but the other square matrices (B, C and D) have no special properties. When desired we will divide them into symmetric and skew-symmetric parts denoted respectively by the subscripts s and a .

We wish to find a relationship between the amplitude of steady state oscillations and the speed parameter v, and also to determine whether the steady state oscillations are stable. Of course, one may wish to vary another parameter rather than v but the procedure should be basically similar to that which we will describe. It is not to be expected that steady state oscillations will be possible at all values of v . The method to be described is an application of that proposed by Roorda and Nemat-Nasser[1]. The rather simpler approach suggested by Taylor, Bogner and Stanley[2], which is a generalisation to a multi-degree of freedom system of the text book energy-balance method (see *eg* Ref 3, p 100), will appear as a by-product.

Let us assume the equation (2-1) has a periodic solution and let the parameter ω which we have introduced, but not specified, be the frequency of this motion. We will,

as an approximation to the solution, take

$$q = \alpha p(x,\tau) \tag{2-2}$$

where α is an amplitude parameter, and x is a column vector of parameters x_i $(i = 1 \to m)$ which are to be determined so that (2-2) is close to the true solution. p will thus be periodic of period 2π and in particular

$$p(x,0) = p(x,2\pi) \; . \tag{2-3}$$

The existence of a steady state solution implies that no energy is accumulated or dissipated over a complete cycle. This means, writing the left-hand side of (2-1) as $y(q,q',q'')$, that our assumed solution must satisfy

$$\alpha \int_0^{2\pi} \{p'(x,\tau)\}^T y(\alpha p, \alpha p', \alpha p'')\, d\tau = 0 \; . \tag{2-4}$$

The matrices A, E and C_s since they are symmetric, and the matrices B_a and D_a since they are skew-symmetric, will make no contribution to this integral. Equation (2-4) can therefore be rewritten

$$\chi(\alpha,\nu,v,x) \equiv \int_0^{2\pi} \{p'(x,\tau)\}^T \left\{\left(B_s \nu + D_s \frac{\nu}{v}\right)\alpha p' + C_a \alpha p + f(\alpha p, \nu\alpha p') + \frac{1}{v^2}\, g(\alpha p, \nu v \alpha p')\right\} d\tau = 0 \tag{2-5}$$

where we have assumed α is not zero.

To obtain further useful relationships we turn to Hamilton's principle which states that the virtual work done by the generalised forces during any admissible virtual displacements over an arbitrary period of time must be zero. Now the elements of y are the generalised forces, since (2-1) is the equation of those forces to zero, and so

$$\int_{\tau_1}^{\tau_2} \delta q^T y(\alpha p, \alpha p', \alpha p'')\, d\tau = 0 \; . \tag{2-6}$$

But from (2-2) we can write δq in terms of arbitrary variations of the elements of x and of the frequency ω. Thus, remembering that $d\tau/d\omega = t = \tau/\omega$,

$$\delta q = \frac{\alpha\tau}{\omega}\, p'(x,\tau)\,\delta\omega + \alpha\left[\frac{\partial p}{\partial x_1}\; \frac{\partial p}{\partial x_2} \;\cdots\; \frac{\partial p}{\partial x_m}\right]\delta x \; . \tag{2-7}$$

Consequently, since $\delta\omega$ and δx are arbitrary, and taking the time interval $(\tau_1 \to \tau_2)$ to be a complete cycle, we have the set of equations

$$\int_0^{2\pi} \{p'(x,\tau)\}^T y(\alpha p, \alpha p', \alpha p'')\tau d\tau = 0 \qquad (2\text{-}8)$$

and

$$\int_0^{2\pi} \left(\frac{\partial p}{\partial x_i}\right)^T y(\alpha p, \alpha p', \alpha p'') d\tau = 0 \qquad i = 1 \to m \ . \qquad (2\text{-}9)$$

We have here assumed that α/ω , as well as α , is not zero.

The above equations (2-5), (2-8) and (2-9) provide a means of determining all but one of the unknowns α, ν, v, x in terms of the remaining one - say determine ν, v and the x_i in terms of the amplitude parameter α . The accuracy of the approximate solution will depend on how adequately the assumed form for the mode of displacement $p(x,\tau)$ can represent the true solution. The method of imposed disturbances[2] (or energy-balance method (Ref 3, p 100)) uses just equation (2-5); the only unknown allowed in the assumed displacement is the amplitude parameter α , the frequency and displacement mode being determined from the *limiting* (linear) case when $\alpha \to 0$. An alternative simplification would be to just use equations (2-5) and (2-8) and so determine say the frequency and speed parameters in terms of the amplitude parameter. However, as the example of section 2.1 shows, it is questionable whether this would yield any improvement over just using (2-5), though it does avoid the assumption that the frequency is unchanged from the linear case.

Before further consideration of how this suggested method could be implemented we will in the next section consider a very simple example in order to get a feel for what is involved.

2.1 A simple, one degree of freedom, example

Consider the equation

$$\nu^2 x'' + \frac{\varepsilon\nu}{v}(x^2 - v)x' + \frac{x}{v^2} = 0 \ . \qquad (2\text{-}10)$$

This represents a one degree of freedom system in which the structural damping is the nonlinear term $\varepsilon\nu x^2 x'/v$, and the only aerodynamic term is the damping $(-\varepsilon\nu x')$. By a change of variables this can be transformed into the standard van der Pol equation

$$\frac{d^2\xi}{d\eta^2} + \mu(\xi^2 - 1)\frac{d\xi}{d\eta} + \xi = 0 \qquad (2\text{-}11)$$

and so we see (*cf* Ref 3, pp 102-106) that, for ε small, equation (2-10) has a limit cycle solution

$$x = 2\sqrt{v}\cos\tau + O(\varepsilon) \tag{2-12}$$

with

$$\nu = \frac{1}{v} + O(\varepsilon^2) \quad . \tag{2-13}$$

Note that the solution of van der Pol's equation transformed to our variables gives $x = 2\sqrt{v}\cos\left\{\frac{1 + O(\varepsilon^2)}{\nu v}\tau\right\} + O(\varepsilon)$, but our choice of ω as the frequency of the periodic solution means that the period in τ is 2π and so we obtain (2-13).

Now if we neglect the nonlinear term in (2-10) we find that the only periodic solution occurs when $v = 0$ and νv is finite and is (α is arbitrary)

$$x = \alpha\cos\left(\frac{\tau}{\nu v}\right) . \tag{2-14}$$

With the proviso about ω this becomes

$$x = \alpha\cos\tau \tag{2-15}$$

with

$$\nu = \frac{1}{v} \quad . \tag{2-16}$$

We will therefore use equation (2-15) as the trial solution, corresponding to equation (2-2), for the methods proposed in section 2 when either just equation (2-5) or equations (2-5) and (2-8) are used. Equation (2-5) is

$$\frac{\alpha\varepsilon\nu}{v}\int_0^{2\pi} (\alpha^2\cos^2\tau - v)\sin^2\tau \, d\tau = 0 \tag{2-17}$$

which gives

$$v = \frac{\alpha^2}{4} \quad . \tag{2-18}$$

In addition for this system equation (2-8) becomes

$$-\alpha\int_0^{2\pi}\left\{-\nu^2\cos\tau - \frac{\varepsilon\nu}{v}(\alpha^2\cos^2\tau - v)\sin\tau + \frac{\cos\tau}{v^2}\right\}\tau\sin\tau \, d\tau = 0 \tag{2-19}$$

and so

$$\frac{1}{2}\left(\nu^2 - \frac{1}{v^2}\right) + \frac{\varepsilon\nu\pi}{v}\left(-\frac{\alpha^2}{4} + v\right) = 0 \quad . \tag{2-20}$$

The solution of this equation is

$$\nu = \frac{1}{v} - \pi\varepsilon\left(-\frac{\alpha^2}{4v} + 1\right) + O(\varepsilon^2) \tag{2-21}$$

which combined with (2-18) gives

$$\nu = \frac{1}{v} + O(\varepsilon^2) \quad . \tag{2-22}$$

Thus, in either case the solution thus obtained has an error $O(\varepsilon)$ in x and one of $O(\varepsilon^2)$ in ν (*cf* equations (2-12) and (2-13); for, when only (2-5) used, ν is assumed to be given by the linear solution (2-16).

Making use also of (2-8) avoids the assumption that the frequency is unchanged from the linear case, but otherwise produces no improvement and it is obvious that this must be a consequence of the trial solution (2-15) being too crude.

To obtain a more accurate solution we take as a trial solution equation (2-15) with the addition of a third harmonic, *ie*

$$x = \alpha\left\{\cos\tau + x_1 \cos 3\tau + x_2 \sin 3\tau\right\} \equiv \alpha p \; . \tag{2-23}$$

This yields, omitting second order terms in x_1 and x_2

$$p' = -\sin\tau - 3x_1 \sin 3\tau + 3x_2 \cos 3\tau \tag{2-24}$$

and

$$\begin{aligned} y(\alpha p, \alpha p', \alpha p'') \approx \alpha \Bigg[& \left\{\left(\frac{1}{v^2} - v^2\right)\cos\tau - \frac{\varepsilon\nu}{v}\sin\tau\,(\alpha^2\cos^2\tau - v)\right\} \\ & + x_1\left\{\left(\frac{1}{v^2} - 9v^2\right)\cos 3\tau - \frac{\varepsilon\nu}{v}\left(2\sin\tau\cos\tau\cos 3\tau + 3(\alpha^2\cos^2\tau - v)\sin 3\tau\right)\right\} \\ & + x_2\left\{\left(\frac{1}{v^2} - 9v^2\right)\sin 3\tau - \frac{\varepsilon\nu}{v}\left(2\sin\tau\cos\tau\sin 3\tau - 3(\alpha^2\cos^2\tau - v)\cos 3\tau\right)\right\}\Bigg] \end{aligned}$$

......(2-25)

Thus we obtain from equations (2-5), (2-8) and (2-9) to the same order of accuracy

$$\begin{aligned} \int_0^{2\pi} \Big[& \sin^2\tau\,(\alpha^2\cos^2\tau - v) + x_1\left\{2\sin^2\tau\cos\tau\cos 3\tau + 6\sin\tau\sin 3\tau\,(\alpha^2\cos^2\tau - v)\right\} \\ & + x_2\left\{2\sin^2\tau\cos\tau\sin 3\tau - 6\sin\tau\cos 3\tau\,(\alpha^2\cos^2\tau - v\right\}\Big]d\tau = 0 \, . \end{aligned}$$

......(2-26)

$$\int_0^{2\pi}\Bigg[\Bigg\{\left(\frac{1}{v^2}-\nu^2\right)\sin\tau\cos\tau-\frac{\varepsilon\nu}{v}\sin^2\tau\,(\alpha^2\cos^2\tau-v)\Bigg\}$$

$$+\,x_1\Bigg\{\left(\frac{1}{v^2}-9\nu^2\right)\sin\tau\cos 3\tau+3\left(\frac{1}{v^2}-\nu^2\right)\cos\tau\sin 3\tau$$

$$-\frac{\varepsilon\nu}{v}\left(2\sin^2\tau\cos\tau\cos 3\tau+6\sin\tau\sin 3\tau\,(\alpha^2\cos^2\tau-v)\right)\Bigg\}$$

$$+\,x_2\Bigg\{\left(\frac{1}{v^2}-9\nu^2\right)\sin\tau\sin 3\tau-3\left(\frac{1}{v^2}-\nu^2\right)\cos\tau\cos 3\tau$$

$$-\frac{\varepsilon\nu}{v}\left(2\sin^2\tau\cos\tau\sin 3\tau-6\sin\tau\cos 3\tau\,(\alpha^2\cos^2\tau-v)\right)\Bigg\}\Bigg]\tau d\tau = 0\,. \quad (2\text{-}27)$$

$$\int_0^{2\pi}\Bigg[\left(\frac{1}{v^2}-\nu^2\right)\cos\tau\cos 3\tau-\frac{\varepsilon\nu}{v}\sin\tau\cos 3\tau\,(\alpha^2\cos^2\tau-v)$$

$$+\,x_1\Bigg\{\left(\frac{1}{v^2}-9\nu^2\right)\cos^2 3\tau-\frac{\varepsilon\nu}{v}\left(2\sin\tau\cos\tau\cos^2 3\tau+3(\alpha^2\cos^2\tau-v)\sin 3\tau\cos 3\tau\right)\Bigg\}$$

$$+\,x_2\Bigg\{\left(\frac{1}{v^2}-9\nu^2\right)\sin 3\tau\cos 3\tau-\frac{\varepsilon\nu}{v}\Big(2\sin\tau\cos\tau\sin 3\tau\cos 3\tau$$

$$-3(\alpha^2\cos^2\tau-v)\cos^2 3\tau\Big)\Bigg\}\Bigg]d\tau = 0\,. \quad (2\text{-}28)$$

and

$$\int_0^{2\pi}\Bigg[\left(\frac{1}{v^2}-\nu^2\right)\cos\tau\sin 3\tau-\frac{\varepsilon\nu}{v}\sin\tau\sin 3\tau\,(\alpha^2\cos^2\tau-v)$$

$$+\,x_1\Bigg\{\left(\frac{1}{v^2}-9\nu^2\right)\sin 3\tau\cos 3\tau-\frac{\varepsilon\nu}{v}\Big(2\sin\tau\cos\tau\sin 3\tau\cos 3\tau$$

$$+3(\alpha^2\cos^2\tau-v)\sin^2 3\tau\Big)\Bigg\}$$

$$+\,x_2\Bigg\{\left(\frac{1}{v^2}-9\nu^2\right)\sin^2 3\tau-\frac{\varepsilon\nu}{v}\Big(2\sin\tau\cos\tau\sin^2 3\tau$$

$$-3(\alpha^2\cos^2\tau-v)\sin 3\tau\cos 3\tau\Big)\Bigg\}\Bigg]d\tau = 0\,. \quad (2\text{-}29)$$

Performing the integrations we get respectively

$$\left(\frac{\alpha^2}{4} - v\right) + x_1\left(-\frac{1}{2} + \frac{6\alpha^2}{4}\right) = 0 \tag{2-30}$$

$$-\frac{1}{2}\left(\frac{1}{v^2} - \nu^2\right) - \frac{\pi\varepsilon\nu}{v}\left(\frac{\alpha^2}{4} - v\right) + x_1\left\{-\frac{2}{v^2} + \frac{\pi\varepsilon\nu}{v}\left(\frac{3\alpha^2 - 1}{2}\right)\right\} + x_2\left\{-\frac{\varepsilon\nu}{v}\left(\frac{7}{24} + \frac{\alpha^2}{8} + \frac{3v}{2}\right)\right\} = 0$$

......(2-31)

$$\left(\frac{1}{v^2} - 9\nu^2\right)x_1 + \frac{3\varepsilon\nu}{v}\left(\frac{\alpha^2}{2} - v\right)x_2 = 0 \tag{2-32}$$

and

$$-\frac{\varepsilon\nu\alpha^2}{4v} - \frac{3\varepsilon\nu}{v}\left(\frac{\alpha^2}{2} - v\right)x_1 + \left(\frac{1}{v^2} - 9\nu^2\right)x_2 = 0 \ . \tag{2-33}$$

The solution of these equations is

$$x_1 = -\frac{3\alpha 4}{64}\varepsilon^2 + 0(\varepsilon^3) \tag{2-34}$$

$$x_2 = -\frac{\alpha^2}{32}\varepsilon + 0(\varepsilon^3) \tag{2-35}$$

$$v = \frac{\alpha^2}{4} + \frac{3\alpha^4(3\alpha^2 - 1)}{128}\varepsilon^2 + 0(\varepsilon^3) \tag{2-36}$$

$$\nu = \frac{4}{\alpha^2} - \frac{\alpha^2}{8}\left(3 + \frac{7\alpha^2}{768} + \frac{\alpha^4}{64}\right)\varepsilon^2 + 0(\varepsilon^3) \tag{2-37}$$

and so from (2-23)

$$x = \alpha \cos\tau - \frac{\alpha^3\varepsilon}{32}\sin 3\tau + 0(\varepsilon^2) \ . \tag{2-38}$$

Comparing with the known solution (*eg* McLachlan[4], p 45) it is easily seen* that we have now got a solution whose error in both x and ν is of $0(\varepsilon^2)$.

3 THE METHOD OF IMPOSED DISTURBANCES (ALIAS ENERGY BALANCE)

When only equation (2-5) is used we take in the trial solution (2-2)

$$p(x,\tau) = \xi\cos\tau - \eta\sin\tau \tag{3-1}$$

* Probably the easiest w[illegible] to do thi[illegible] is to put $v = 1$ and then, from (2-36), $\alpha^2 = 4\left(1 - \frac{33}{8}\varepsilon^2\right)$. I[illegible] the coefficient of ε^2 in the expression for ν (equation (2-37)) is (953/[illegible]) compared with the correct value of (−1/16).

where $(\xi + i\eta)$ is the right hand eigenvector satisfying

$$\left\{ - A\nu_\ell^2 + i\left(B + \frac{D}{v_\ell}\right)\nu_\ell + \left(C + \frac{E}{v_\ell^2}\right) \right\} (\xi + i\eta) = 0 \tag{3-2}$$

in the critical state when ν_ℓ and v_ℓ are real. Thus ν_ℓ, v_ℓ and $(\xi + i\eta)$ are obtained by the solution of the normal linear flutter problem. Two elements, all told, in ξ and η will be arbitrary. We also assume that the frequency in the nonlinear case is the same as in the linear case, *ie* we take

$$\nu v = \nu_\ell v_\ell \quad . \tag{3-3}$$

Other assumptions could be made but in the absence of any strong evidence the above seems a reasonable choice. Substituting from (3-1) and (3-3) in (2-5) and integrating the linear terms, then gives the following equation relating the amplitude α of maintained oscillations with the speed parameter v .

$$\alpha\pi \left\{ \nu_\ell v_\ell \left[\xi^T \left(\frac{1}{v} B_s + \frac{1}{v^2} D_s \right) \xi + \eta^T \left(\frac{1}{v} B_s + \frac{1}{v^2} D_s \right) \eta \right] + 2\xi^T C_a \eta \right\}$$

$$= \int_0^{2\pi} (\xi^T \sin\tau + \eta^T \cos\tau) \left\{ f\left[\alpha(\xi\cos\tau - \eta\sin\tau), \frac{\alpha\nu_\ell v_\ell}{v} (-\xi\sin\tau - \eta\cos\tau) \right] + \frac{1}{v^2} g\left[\alpha(\xi\cos\tau - \eta\sin\tau), \alpha\nu_\ell v_\ell(-\xi\sin\tau - \eta\cos\tau) \right] \right\} d\tau .$$

......(3-4)

Alternatively it could have been written as an equation connecting α and the frequency parameter ν . If the nonlinear aerodynamic and structural terms, f and g , can be evaluated numerically with little difficulty - one might, for example, have an analytical approximation to g deduced from resonance tests - then the solution of (3-4) should be no problem. However, one may obtain f or g by a timewise integration procedure. Thus one would have to assume values of α and v (or equivalently ν), evaluate the right hand side of (3-4) and then compare it with the left, and then repeat for other (α, v) until one had got equality. In such a case a scalar parameter could be put on, say, D_s (the symmetric part of the structural damping matrix) and the results interpreted as values of this parameter as a function of α and v . The curve in the (α, v) plane where the parameter was unity would then indicate the critical states - the limit cycles.

3.1 Energy balance with frequency determination

In this section we consider in more detail the use of (2-8) along with (2-5). In the trial solution $p(x,\tau)$ is again taken to be given by equations (3-1) and (3-2). The assumption (3-3) is, however, no longer needed. Thus (2-5) gives

$$\alpha\pi\left\{\nu\xi^T\left(B_s + \frac{1}{\nu} D_s\right)\xi + \nu\eta^T\left(B_s + \frac{1}{\nu} D_s\right)\eta + 2\xi^T C_a\eta\right\}$$

$$= \int_0^{2\pi} \left(\xi^T \sin\tau + \eta^T \cos\tau\right)\left\{f\left[\alpha(\xi\cos\tau - \eta\sin\tau),\ \alpha\nu(-\xi\sin\tau - \eta\cos\tau)\right] + \frac{1}{\nu^2}\ \times\right.$$

$$\left.\times\, g\left[\alpha(\xi\cos\tau - \eta\sin\tau), \alpha\nu\nu(-\xi\sin\tau - \eta\cos\tau)\right]\right\}d\tau \qquad (3\text{-}5)$$

while from (2-8) we have

$$\alpha\left\{\xi^T\left(2\pi^2 C_a - \pi\nu\left[B_s + \frac{1}{\nu} D_s\right]\right)\eta + \xi^T\left(\pi^2\nu\left[B_s + \frac{1}{\nu} D_s\right] + \frac{\pi}{2}\left[C_s + \frac{1}{\nu^2} E - \nu^2 A\right]\right)\xi\right.$$

$$\left. + \eta^T\left(\pi^2\nu\left[B_s + \frac{1}{\nu} D_s\right] - \frac{\pi}{2}\left[C_s + \frac{1}{\nu^2} E - \nu^2 A\right]\right)\eta\right\}$$

$$= \int_0^{2\pi} (\xi^T \sin\tau + \eta^T \cos\tau)\left\{f\left[\alpha(\xi\cos\tau - \eta\sin\tau), \alpha\nu(-\xi\sin\tau - \eta\cos\tau)\right]\right.$$

$$\left. + \frac{1}{\nu^2}\, g\left[\alpha(\xi\cos\tau - \eta\sin\tau), \alpha\nu\nu(-\xi\sin\tau - \eta\cos\tau)\right]\right\}\tau d\tau\,.$$

......(3-6)

These two equations are particularly convenient in the case when the linear structural damping (or indeed just the symmetrical part of the linear structural damping matrix) is zero and there is no structural nonlinearity. It is common practice to assume no structural damping and rely on what there is present in practice for an extra little bit of safety margin*. In this particular case D_s and g will be zero therefore. The first equation (3-5), which then becomes

$$\alpha\pi\left\{\nu\left(\xi^T B_s\xi + \eta^T B_s\eta\right) + 2\xi^T C_a\eta\right\}$$

$$= \int_0^{2\pi} (\xi^T \sin\tau + \eta^T \cos\tau) f\left[\alpha(\xi\cos\tau - \eta\sin\tau), \alpha\nu(-\xi\sin\tau - \eta\cos\tau)\right] d\tau \qquad (3\text{-}7)$$

can be solved, on the lines discussed in section 3, for ν as a function of α.

* The addition of structural damping does not, however, necessarily make a system more stable.

The speed parameter v is then immediately evaluated from equation (3-6) which in this particular case gives

$$\frac{1}{v^2} = \frac{1}{(\xi^T E\xi - \eta^T E\eta)}\Bigg\{\xi^T\left(\nu^2 A - 2\pi\nu B_s - C_s\right)\xi - \eta^T\left(\nu^2 A + 2\pi\nu B_s - C_s\right)\eta$$

$$+ \frac{2}{\pi\alpha}\int_0^{2\pi}(\xi^T \sin\tau + \eta^T \cos\tau) f\Big[\alpha(\xi\cos\tau - \eta\sin\tau),$$

$$\alpha\nu(-\xi\sin\tau - \eta\cos\tau)\Big]\tau\, d\tau\Bigg\} . \quad (3\text{-}8)$$

When the only nonlinearity is structural the solution is in one sense rather more difficult. We can eliminate v between the two equations, (3-5) and (3-6), and obtain an equation connecting α and νv. Thus writing the following four constants as

$$k_1 = \xi^T B_s \xi + \eta^T B_s \eta \quad (3\text{-}9)$$

$$k_2 = \xi^T B_s \xi + \eta^T B_s \eta - \frac{1}{\pi}\xi^T B_s \eta \quad (3\text{-}10)$$

$$c_1 = 2\xi^T C_a \eta \quad (3\text{-}11)$$

$$c_2 = 2\xi^T C_a \eta + \frac{1}{2\pi}\left\{\xi^T C_s \xi - \eta^T C_s \eta\right\} \quad (3\text{-}12)$$

and also using the notation

$$g_1(\alpha,\nu v) = \frac{1}{\alpha(\nu v)^2\pi}\int_0^{2\pi}(\xi^T \sin\tau + \eta^T \cos\tau) g\Big[\alpha(\xi\cos\tau - \eta\sin\tau),$$

$$\alpha\nu v(-\xi\sin\tau - \eta\cos\tau)\Big] d\tau$$

$$- \frac{1}{\nu v}\left(\xi^T D_s \xi + \eta^T D_s \eta\right) \quad (3\text{-}13)$$

$$g_2(\alpha,\nu v) = \frac{1}{\alpha(\nu v)^2\pi^2}\int_0^{2\pi}(\xi^T \sin\tau + \eta^T \cos\tau) g\Big[\alpha(\xi\cos\tau - \eta\sin\tau),$$

$$\alpha\nu v(-\xi\sin\tau - \eta\cos\tau)\Big]\tau\, d\tau$$

$$- \frac{1}{2\pi(\nu v)^2}(\xi^T E\xi - \eta^T E\eta) - \frac{1}{\nu v}\left(\xi^T D_s \xi + \eta^T D_s \eta - \frac{1}{\pi}\xi^T D_s \eta\right) + \frac{1}{2\pi}\left(\xi^T A\xi - \eta^T A\eta\right)$$

$$\ldots\ldots(3\text{-}14)$$

we obtain the relationship

$$(c_1 k_2 - c_2 k_1)\left\{k_2 g_1(\alpha,\nu v) - k_1 g_2(\alpha,\nu v)\right\} = \left\{c_2 g_1(\alpha,\nu v) - c_1 g_2(\alpha,\nu v)\right\}^2 . \quad (3\text{-}15)$$

Having solved this equation for νv as a function of α one can then return to equation (3-5) or (3-6) to determine ν and hence v. In this case (no aerodynamic nonlinearity), with the notation introduced in equations (3-9) to (3-14), they become

$$\begin{cases} \nu^2 g_1(\alpha,\nu v) - \nu k_1 - c_1 = 0 & (3\text{-}16) \\ \nu^2 g_2(\alpha,\nu v) - \nu k_2 - c_2 = 0 \; . & (3\text{-}17) \end{cases}$$

4 THE FULL PROPOSAL

Our full proposal involving the use of equations (2-5), (2-8) and (2-9) permits the introduction of some unknown parameters x_r - the elements of x - in the mode $p(x,\tau)$ of the trial solution (2-2). In the one degree of freedom example considered in section (2-1) these were taken to be coefficients of harmonics higher than the fundamental (*cf* equation (2-23). For the multi-degree of freedom system there is more choice. One could alternatively allow variation in the relative values of the elements of $p(x,\tau)$ rather than introduce higher harmonics, or perhaps use some mixture of the two ideas.

Baldock[5,6] has demonstrated how one can often find an equivalent two degree of freedom system which will represent the behaviour of a system with many more degrees of freedom with good accuracy at or in the vicinity of the flutter condition. Let the modes of these two degrees of freedom be represented in terms of the original generalised coordinates by the column vectors ϕ and θ, and let the flutter mode of the linear system be

$$q = (z_1 + i)\phi + (1 + iz_2)\theta \; . \qquad (4\text{-}1)$$

Then, on the basis of this, it seems reasonable to take as a trial solution for the nonlinear system

$$q = \alpha p(x,\tau) = \alpha\Big[\Big\{(z_1 + x_1)\phi + \theta\Big\} \cos \tau - \Big\{\phi + (z_2 + x_2)\theta\Big\} \sin \tau\Big] \; . \qquad (4\text{-}2)$$

Substituting this expression in equations (2-5) and (2-8) gives the equations given in the previous section (3-1) - equations (3-5) and (3-6) in the general case - with

$$\begin{cases} \xi = (z_1 + x_1)\phi + \theta & (4\text{-}3) \\ \eta = \phi + (z_2 + x_2)\theta \; . & (4\text{-}4) \end{cases}$$

Of course, we only wish to retain first order terms in x_1 and x_2, and so writing

$$\xi_0 = (\xi)_{x_1=0} \qquad (4\text{-}5)$$

$$\eta_0 = (\eta)_{x_2=0} \qquad (4\text{-}6)$$

$$f_0(\alpha,\nu,\tau) = f\left[\alpha(\xi_0 \cos\tau - \eta_0 \sin\tau), \alpha\nu(-\xi_0 \sin\tau - \eta_0 \cos\tau)\right] \tag{4-7}$$

and

$$g_0(\alpha,\nu v,\tau) = g\left[\alpha(\xi_0 \cos\tau - \eta_0 \sin\tau), \alpha\nu v(-\xi_0 \sin\tau - \eta_0 \cos\tau)\right] \tag{4-8}$$

equation (3-5) becomes

$$\begin{aligned}
&\alpha\pi\Big\{\nu\xi_0^T\Big(B_s + \frac{1}{v}D_s\Big)\xi_0 + \nu\eta_0^T\Big(B_s + \frac{1}{v}D_s\Big)\eta_0 + 2\xi_0^T C_a\eta_0 + 2x_1\Big[\nu\xi_0^T\Big(B_s + \frac{1}{v}D_s\Big)\phi - \eta_0^T C_a\phi\Big] \\
&\qquad + 2x_2\Big[\nu\eta_0^T\Big(B_s + \frac{1}{v}D_s\Big)\theta + \xi_0^T C_a\theta\Big]\Big\} \\
&= \int_0^{2\pi}\Big(\xi_0^T \sin\tau + \eta_0^T \cos\tau\Big)\Big\{f_0(\alpha,\nu,\tau) + \frac{1}{v^2} g_0(\alpha,\nu v,\tau)\Big\}d\tau \\
&+ x_1\int_0^{2\pi}\Big[\phi^T \sin\tau\, f_0(\alpha,\nu,\tau) + \Big(\xi_0^T \sin\tau + \eta_0^T \cos\tau\Big) \times \\
&\qquad \times\Big\{F_0^{(q)}(\alpha,\nu,\tau)\phi\alpha\cos\tau - F_0^{(q')}(\alpha,\nu,\tau)\phi\alpha\sin\tau\Big\}\Big]d\tau \\
&+ x_2\int_0^{2\pi}\Big[\theta^T \cos\tau\, f_0(\alpha,\nu,\tau) - \Big(\xi_0^T \sin\tau + \eta_0^T \cos\tau\Big) \times \\
&\qquad \times\Big\{F_0^{(q)}(\alpha,\nu,\tau)\theta\alpha\sin\tau + F_0^{(q')}(\alpha,\nu,\tau)\theta\alpha\cos\tau\Big\}\Big]d\tau \\
&+ \frac{x_1}{v^2}\int_0^{2\pi}\Big[\phi^T \sin\tau\, g_0(\alpha,\nu v,\tau) + \Big(\xi_0^T \sin\tau + \eta_0^T \cos\tau\Big) \times \\
&\qquad \times\Big\{G_0^{(q)}(\alpha,\nu v,\tau)\phi\alpha\cos\tau - G_0^{(q')}(\alpha,\nu v,\tau)\phi\alpha\sin\tau\Big\}\Big]d\tau \\
&+ \frac{x_2}{v^2}\int_0^{2\pi}\Big[\theta^T \cos\tau\, g_0(\alpha,\nu v,\tau) - \Big(\xi_0^T \sin\tau + \eta_0^T \cos\tau\Big) \times \\
&\qquad \times\Big\{G_0^{(q)}(\alpha,\nu v,\tau)\theta\alpha\sin\tau + G_0^{(q')}(\alpha,\nu v,\tau)\theta\alpha\cos\tau\Big\}\Big]d\tau
\end{aligned} \tag{4-9}$$

where (*cf* section 4.1) $F_0^{(q)}$, $F_0^{(q')}$, $G_0^{(q)}$, $G_0^{(q')}$ are square matrices whose jth columns are respectively

$$\left.\begin{array}{l} \left\{\dfrac{\partial f(q,\nu q')}{\partial q_j}\right\}_{q=\alpha(\xi_0\ \cos\ \tau-\eta_0\ \sin\ \tau)} \\[2ex] \left\{\dfrac{\partial f(q,\nu q')}{\partial q_j'}\right\}_{q=\alpha(\xi_0\ \cos\ \tau-\eta_0\ \sin\ \tau)} \\[2ex] \left\{\dfrac{\partial g(q,\nu v q')}{\partial q_j}\right\}_{q=\alpha(\xi_0\ \cos\ \tau-\eta_0\ \sin\ \tau)} \\[2ex] \text{and} \quad \left\{\dfrac{\partial g(q,\nu v q')}{\partial q_j'}\right\}_{q=\alpha(\xi_0\ \cos\ \tau-\eta_0\ \sin\ \tau)} \end{array}\right\} \tag{4-10}$$

q_j being the jth element of q . We are assuming that the functions f and g have a Taylor expansion at this point. Similarly equation (3-6) becomes

$$\begin{aligned}\alpha\Big\{\xi_0^T\Big(2\pi^2C_a - \pi\nu\Big[B_s + \frac{1}{v}D_s\Big]\Big)\eta_0 &+ \xi_0^T\Big(\pi^2\nu\Big[B_s + \frac{1}{v}D_s\Big] + \frac{\pi}{2}\Big[C_s + \frac{1}{v^2}E - \nu^2A\Big]\Big)\xi_0 \\ &+ \eta_0^T\Big(\pi^2\nu\Big[B_s + \frac{1}{v}D_s\Big] - \frac{\pi}{2}\Big[C_s + \frac{1}{v^2}E - \nu^2A\Big]\Big)\eta_0 \\ &+ x_1\Big[\nu\Big(\pi\eta_0^T + 2\pi^2\xi_0^T\Big)\Big(B_s + \frac{1}{v}D_s\Big) - 2\pi^2\eta_0^TC_a + \pi\xi_0^T\Big(C_s + \frac{1}{v^2}E - \nu^2A\Big)\Big]\phi \\ &- x_2\Big[\nu\Big(\pi\xi_0^T - 2\pi^2\eta_0^T\Big)\Big(B_s + \frac{1}{v}D_s\Big) - 2\pi^2\xi_0^TC_a + \pi\eta_0^T\Big(C_s + \frac{1}{v^2}E - \nu^2A\Big)\Big]\theta\Big\}\end{aligned}$$

$$\begin{aligned} = &\int_0^{2\pi}\Big(\xi_0^T\ \sin\ \tau + \eta_0^T\ \cos\ \tau\Big)\Big\{f_0(\alpha,\nu,\tau) + \frac{1}{v^2}\,g_0(\alpha,\nu v,\tau)\Big\}\tau d\tau \\ &+ x_1\int_0^{2\pi}\Big[\phi^T\ \sin\ \tau\ f_0(\alpha,\nu,\tau) + \Big(\xi_0^T\ \sin\ \tau + \eta_0^T\ \cos\ \tau\Big)\ \times \\ &\qquad \times\Big\{F_0^{(q)}(\alpha,\nu,\tau)\phi\alpha\ \cos\ \tau - F_0^{(q')}(\alpha,\nu,\tau)\phi\alpha\ \sin\ \tau\Big\}\Big]\tau d\tau \\ &+ x_2\int_0^{2\pi}\Big[\theta^T\ \cos\ \tau\ f_0(\alpha,\nu,\tau) - \Big(\xi_0^T\ \sin\ \tau + \eta_0^T\ \cos\ \tau\Big)\ \times \\ &\qquad \times\Big\{F_0^{(q)}(\alpha,\nu,\tau)\theta\alpha\ \sin\ \tau + F_0^{(q')}(\alpha,\nu,\tau)\theta\alpha\ \cos\ \tau\Big\}\Big]\tau d\tau \\ &+ \frac{x_1}{v^2}\int_0^{2\pi}\Big[\phi^T\ \sin\ \tau\ g_0(\alpha,\nu v,\tau) + \Big(\xi_0^T\ \sin\ \tau + \eta_0^T\ \cos\ \tau\Big)\ \times \\ &\qquad \times\Big\{G_0^{(q)}(\alpha,\nu v,\tau)\phi\alpha\ \cos\ \tau - G_0^{(q')}(\alpha,\nu v,\tau)\phi\alpha\ \sin\ \tau\Big\}\Big]\tau d\tau \end{aligned}$$

$$+ \frac{x_2}{v^2} \int_0^{2\pi} \left[\theta^T \cos\tau\, g_0(\alpha,\nu v,\tau) - \left(\xi_0^T \sin\tau + \eta_0^T \cos\tau\right) \times \right.$$
$$\left. \times \left\{G_0^{(q)}(\alpha,\nu v,\tau)\theta\alpha \sin\tau + G_0^{(q')}(\alpha,\nu v,\tau)\theta\alpha \cos\tau\right\}\right]\tau d\tau \quad . \qquad (4\text{-}11)$$

Furthermore, from equation (2-9), with $p(x,\tau)$ given by (4-2), we obtain the following two equations, keeping again only first order terms in x_1 and x_2 :

$$\alpha\pi\phi^T\left[\left\{\left(C + \frac{E}{v^2} - A\nu^2\right)\xi_0 - \nu\left(B + \frac{D}{v}\right)\eta_0\right\} + x_1\left(C_s + \frac{E}{v^2} - A\nu^2\right)\phi - \nu x_2\left(B + \frac{D}{v}\right)\theta\right]$$
$$= -\phi^T \int_0^{2\pi} \left[\left\{f_0(\alpha,\nu,\tau) + \frac{1}{v^2} g_0(\alpha,\nu v,\tau)\right\}\cos\tau\right.$$
$$+ \alpha x_1\left\{F_0^{(q)}(\alpha,\nu,\tau)\phi \cos^2\tau - F_0^{(q')}(\alpha,\nu,\tau)\phi \sin\tau\cos\tau\right\}$$
$$+ \frac{\alpha x_1}{v^2}\left\{G_0^{(q)}(\alpha,\nu v,\tau)\phi \cos^2\tau - G_0^{(q')}(\alpha,\nu v,\tau)\phi \sin\tau\cos\tau\right\}$$
$$- \alpha x_2\left\{F_0^{(q)}(\alpha,\nu,\tau)\theta \sin\tau\cos\tau + F_0^{(q')}(\alpha,\nu,\tau)\theta \cos^2\tau\right\}$$
$$\left. - \frac{\alpha x_2}{v^2}\left\{G_0^{(q)}(\alpha,\nu v,\tau)\theta \sin\tau\cos\tau + G_0^{(q')}(\alpha,\nu v,\tau)\theta \cos^2\tau\right\}\right] d\tau$$
$$\ldots\ldots(4\text{-}12)$$

and

$$\alpha\pi\theta^T\left[\left\{\left(C + \frac{E}{v^2} - A\nu^2\right)\eta_0 + \nu\left(B + \frac{D}{v}\right)\xi_0\right\} + \nu x_1\left(B + \frac{D}{v}\right)\phi + x_2\left(C_s + \frac{E}{v^2} - A\nu^2\right)\theta\right]$$
$$= \theta^T \int_0^{2\pi} \left[\left\{f_0(\alpha,\nu,\tau) + \frac{1}{v^2} g_0(\alpha,\nu v,\tau)\right\}\sin\tau\right.$$
$$+ \alpha x_1\left\{F_0^{(q)}(\alpha,\nu,\tau)\phi \sin\tau\cos\tau - F_0^{(q')}(\alpha,\nu,\tau)\phi \sin^2\tau\right\}$$
$$+ \frac{\alpha x_1}{v^2}\left\{G_0^{(q)}(\alpha,\nu v,\tau)\phi \sin\tau\cos\tau - G_0^{(q')}(\alpha,\nu v,\tau)\phi \sin^2\tau\right\}$$
$$- \alpha x_2\left\{F_0^{(q)}(\alpha,\nu,\tau)\theta \sin^2\tau + F_0^{(q')}(\alpha,\nu,\tau)\theta \sin\tau\cos\tau\right\}$$
$$\left. - \frac{\alpha x_2}{v^2}\left\{G_0^{(q)}(\alpha,\nu v,\tau)\theta \sin^2\tau + G_0^{(q')}(\alpha,\nu v,\tau)\theta \sin\tau\cos\tau\right\}\right] d\tau \quad .$$
$$\ldots\ldots(4\text{-}13)$$

Thus we have four equations, (4-9), (4-11), (4-12) and (4-13), which can be solved for the four of the five unknowns α, ν, v, x_1 and x_2 in terms of the other one. The equations are linear in the modal parameters x_1 and x_2 but not in the other unknowns.

The solution of these equations should not be as troublesome as it might appear. Consider first the case when there is no structural damping and no structural non-linearity (D and $g(q,\nu v q')$ are both zero). Then all the integrals that appear are independent of v and so we can assume values of α and ν and obtain respectively, from equations (4-9), (4-11), (4-12) and (4-13), equation of the form

$$\mu_{01}x_1 + \mu_{02}x_2 + \mu_{03} = 0 \qquad (4\text{-}14)$$

$$\left(\mu_{11} - \frac{\alpha}{v^2}\gamma_{11}\right)x_1 + \left(\mu_{12} - \frac{\alpha}{v^2}\gamma_{12}\right)x_2 + \left(\mu_{13} - \frac{\alpha}{v^2}\gamma_{13}\right) = 0 \qquad (4\text{-}15)$$

$$\left(\mu_{21} - \frac{\alpha}{v^2}\gamma_{21}\right)x_1 + \mu_{22}x_2 + \left(\mu_{23} - \frac{\alpha}{v^2}\gamma_{23}\right) = 0 \qquad (4\text{-}16)$$

$$\mu_{31}x_1 + \left(\mu_{32} - \frac{\alpha}{v^2}\gamma_{32}\right)x_2 + \left(\mu_{33} - \frac{\alpha}{v^2}\gamma_{33}\right) = 0\ . \qquad (4\text{-}17)$$

The coefficients μ_{ij} will all be functions of α and ν . For example

$$\mu_{02} = 2\alpha\pi\left[\nu\eta_0^T B_s\theta + \xi_0^T C_a\theta\right] - \int_0^{2\pi}\left[\theta^T \cos\tau\ f_0(\alpha,\nu,\tau) - \left(\xi_0^T \sin\tau + \eta_0^T \cos\tau\right) \times \right.$$
$$\left. \times \left\{F_0^{(q)}(\alpha,\nu,\tau)\theta\alpha \sin\tau - F_0^{(q')}(\alpha,\nu,\tau)\theta\alpha\cos\tau\right\}\right]d\tau\ . \qquad (4\text{-}18)$$

The other coefficients, the γ_{ij} , are constants. An example is

$$\gamma_{32} = -\pi\theta^T E\theta\ . \qquad (4\text{-}19)$$

The three equations (4-15) to (4-17) form an eigenvalue problem which can be solved by the usual methods for α/v^2, x_1 and x_2 . We require a solution for which all these quantities are real and the first one, (α/v^2), is positive. There always will be at least one real solution. The values of x_1 and x_2 , from suitable solutions of (4-15) to (4-17), can then be substituted in equation (4-14) to see how closely it is satisfied. By repeating the procedure for other assumed values of α and ν one should then be able to find the conditions under which all four equations are satisfied; thus obtaining v, x_1, x_2, and ν as functions of α for the critical (limit cycle) flutter condition.

When the only nonlinearity is structural it is convenient to use a notation which is an elaboration of that used in that section 3.1 (see Table 1 for full details). Equations (4-9), (4-11), (4-12) and (4-13) then become

- a factor $\alpha\pi$ or $\alpha\pi^2$ has been deleted as convenient
- respectively

$$\nu^2\left\{g_{10}(\alpha,\nu v) + x_1 g_{11}(\alpha,\nu v) + x_2 g_{12}(\alpha,\nu v)\right\} - \nu(k_{10} + k_{11}x_1 + k_{12}x_2)$$
$$- (c_{10} - c_{11}x_1 + c_{12}x_2) = 0 \qquad (4\text{-}20)$$

$$\nu^2\left\{g_{20}(\alpha,\nu v) + x_1 g_{21}(\alpha,\nu v) + x_2 g_{22}(\alpha,\nu v)\right\} - \nu(k_{20} + k_{21}x_1 + k_{22}x_2)$$
$$- (c_{20} - c_{21}x_1 + c_{22}x_2) = 0 \qquad (4\text{-}21)$$

$$\nu^2\left\{g_{30}(\alpha,\nu v) + x_1 g_{31}(\alpha,\nu v) - x_2 g_{32}(\alpha,\nu v)\right\} - \nu(k_{30} + k_{32}x_2) + (c_{30} + c_{31}x_1) = 0 \qquad (4\text{-}22)$$

$$\nu^2\left\{g_{40}(\alpha,\nu v) + x_1 g_{41}(\alpha,\nu v) - x_2 g_{42}(\alpha,\nu v)\right\} - \nu(k_{40} + k_{41}x_1) - (c_{40} + c_{42}x_2) = 0 \quad . \qquad (4\text{-}23)$$

The procedure that can be followed will then be similar to that just suggested in the case of aerodynamic (and not structural) nonlinearity. Values of α and νv assumed; the eigenvalue value problem posed by three* *of the above four equations* is solved for ν^2, x_1 and x_2; the solution is substituted in the remaining equation; and the process is repeated with other assumed values of α and νv until the *locus* of critical condition is found.

4.1 The matrices $F_0^{(q)}$ etc

No difficulty should be encountered in evaluating the matrices whose elements are defined by equations (4-10) if analytical expressions are known for the column vectors $f(q,\nu q')$ *and* $g(q,\nu v q')$. There may, however, be cases when all that is available are numerical values of the vectors at instances during a specified motion. For example this most likely will be the case for f as given by nonlinear transonic flow calculations. At first sight one may imagine in such a case that we are left with an almost overwhelming problem.

However, let us first note that what is required in equations (4-9) and (4-11) to (4-13) is not the two square matrices $F_0^{(q)}$ and $F_0^{(q')}$ in isolation but the two column vectors

$$\left(F_0^{(q)}\phi \cos\tau - F_0^{(q')}\phi \sin\tau\right) = \frac{1}{\alpha}\left(\frac{\partial f}{\partial z_1}\right)_{q=\alpha p_0} \qquad (4\text{-}24)$$

and

$$\left(F_0^{(q)}\theta \sin\tau + F_0^{(q')}\theta \cos\tau\right) = -\frac{1}{\alpha}\left(\frac{\partial f}{\partial z_2}\right)_{q=\alpha p_0} \qquad (4\text{-}25)$$

---

* Stability considerations (*cf* section 4.1) suggest that it may be most instructive to take the last three - (4-21) to (4-23). A solution which gives ν^2 real and positive (and hence x_1 and x_2 real) is required.

where $$p_0(\tau) = p(0,\tau) . \quad (4\text{-}26)$$

Thus by evaluating $f(q,\nu q')$ for $q = \alpha p_0$, and for one or two adjacent q caused by small variations in z_1 and z_2 , one should be able to evaluate the required vectors.

4.2 Stability of limit cycles

When the function $\chi(\alpha,\nu,v,x)$ (equation (2-5)) is positive*, it means that energy has to be supplied to the system to maintain the assumed motion. Our proposed solution procedure determines a curve, in say the (α,ν) plane, where $\chi = 0$; and also provides values of χ at other points in this plane. If χ is negative above this critical curve, *ie* when α is increased by $\delta\alpha$, or positive below the curve, then we can certainly say that the limit cycles corresponding to points on the curve are unstable for they are unstable with respect to perturbations in α . We cannot, however, say with absolute certainty that the limit cycles are stable if the contrary is true, that is if χ is positive above and negative below the critical curve. This is because the motion may be unstable with respect to other perturbations (*cf* Ref 1). However, in practice, one would expect the signs of χ , adjacent to the critical curve $\chi = 0$, to be a good enough guide to the stability or otherwise of the limit cycles. If there is a stable limit cycle then it means that we can have what is usually known as limited amplitude flutter; while if there is just an unstable limit cycle then catastrophic (diverging to infinity) flutter will be possible when the system undergoes a big enough disturbance.

5 CONCLUDING REMARKS

The question as to whether procedures for determining limit cycles, such as those described in this Memorandum, are more attractive than other approaches will depend largely on the number of parameters necessary in the description of the limit cycle. If, as in the detailed development of section 4, one can get away with two parameters x_1 and x_2 , in addition to the amplitude and frequency parameters (α and ν) (*cf* equation (4-2)), then it may well prove the more economical method. Baldock's way[5,6] of condensing a linear flutter system to an equivalent binary certainly provides a promising guide to the choice of parameters in such a case.

* Or, equivalently, the left-hand side is greater than the right-hand side in equations (3-4) or (3-5) or (4-9), or the left-hand side of (4-20) is negative. The energy supplied is also zero in the trivial case $\alpha = 0$ (*cf* equation (2-4)).

Table 1

NOTATION USED IN EQUATIONS (4-20) TO (4-23)

$$c_{10} = 2\xi_0^T C_a \eta_0 \;, \qquad c_{11} = 2\eta_0^T C_a \phi \;, \qquad c_{12} = 2\xi_0^T C_a \theta \;,$$

$$c_{20} = 2\xi_0^T C_a \eta_0 + \frac{1}{2\pi}\left\{\xi_0^T C_s \xi_0 - \eta_0^T C_s \eta_0\right\}, \quad c_{21} = 2\eta_0^T C_a \phi - \frac{1}{\pi}\,\xi_0^T C_s \phi \;, \quad c_{31} = 2\xi_0^T C_a \theta - \frac{1}{\pi}\,\eta_0^T C_s \theta$$

$$c_{30} = \phi^T C \xi_0 \;, \qquad c_{31} = \phi^T C_s \phi \;,$$

$$c_{40} = \theta^T C \eta_0 \;, \qquad c_{42} = \theta^T C_s \theta$$

$$=$$

$$k_{10} = \xi_0^T B_s \xi_0 + \eta_0^T B_s \eta_0 \;, \qquad k_{11} = 2\xi_0^T B_s \phi \;, \qquad k_{12} = 2\eta_0^T B_s \theta \;,$$

$$k_{20} = \xi_0^T B_s \xi_0 + \eta_0^T B_s \eta_0 - \frac{1}{\pi}\,\xi_0^T B_s \eta_0 \;, \quad k_{21} = 2\xi_0^T B_s \phi + \frac{1}{\pi}\,\eta_0^T B_s \phi \;, \quad k_{22} = 2\eta_0^T B_s \theta - \frac{1}{\pi}\,\xi_0^T B_s \theta \;,$$

$$k_{30} = \phi^T B \eta_0 \;, \qquad k_{32} = \phi^T B \theta$$

$$k_{40} = \theta^T B \xi_0 \;, \qquad k_{41} = \theta^T B \phi$$

$$=$$

$$g_{10}(\alpha,\nu v) = \frac{1}{\alpha\pi(\nu v)^2}\int_0^{2\pi}\left(\xi_0^T \sin\tau + \eta_0^T \cos\tau\right) g_0(\alpha,\nu v,\tau)\,d\tau - \frac{1}{\nu v}\left(\xi_0^T D_s \xi_0 + \eta_0^T D_s \eta_0\right)$$

$$g_{11}(\alpha,\nu v) = \frac{1}{\alpha\pi(\nu v)^2}\int_0^{2\pi}\Big[\phi^T \sin\tau\, g_0(\alpha,\nu v,\tau) + \left(\xi_0^T \sin\tau + \eta_0^T \cos\tau\right) \times \\ \times\left\{G_0^{(q)}(\alpha,\nu v,\tau)\phi\alpha\cos\tau - G_0^{(q')}(\alpha,\nu v,\tau)\phi\alpha\sin\tau\right\}\Big]d\tau - \frac{2}{\nu v}\,\xi_0^T D_s \phi$$

$$g_{12}(\alpha,\nu v) = \frac{1}{\alpha\pi(\nu v)^2}\int_0^{2\pi}\Big[\theta^T \cos\tau\, g_0(\alpha,\nu v,\tau) - \left(\xi_0^T \sin\tau + \eta_0^T \cos\tau\right) \times \\ \times\left\{G_0^{(q)}(\alpha,\nu v,\tau)\theta\alpha\sin\tau + G_0^{(q')}(\alpha,\nu v,\tau)\theta\alpha\cos\tau\right\}\Big]d\tau - \frac{2}{\nu v}\,\eta_0^T D_s \theta$$

$$g_{20}(\alpha,\nu v) = \frac{1}{\alpha\pi^2(\nu v)^2}\int_0^{2\pi}\left(\xi_0^T \sin\tau + \eta_0^T \cos\tau\right) g_0(\alpha,\nu v,\tau)\,\tau\,d\tau - \frac{1}{2\pi(\nu v)^2}\left(\xi_0^T E \xi_0 - \eta_0^T E \eta_0\right) \\ - \frac{1}{\nu v}\left(\xi_0^T D_s \xi_0 + \eta_0^T D_s \eta_0 - \frac{1}{\pi}\,\xi_0^T D_s \eta_0\right) + \frac{1}{2\pi}\left(\xi_0^T A \xi_0 - \eta_0^T A \eta_0\right)$$

Table 1 (concluded)

$$g_{21}(\alpha,\nu v) = \frac{1}{\alpha\pi^2(\nu v)^2}\int_0^{2\pi}\Big[\phi^T \sin\tau\, g_0(\alpha,\nu v,\tau) + \left(\xi_0^T \sin\tau + \eta_0^T \cos\tau\right)\times$$
$$\times\left\{G_0^{(q)}(\alpha,\nu v,\tau)\phi\alpha\cos\tau - G_0^{(q')}(\alpha,\nu v,\tau)\phi\alpha\sin\tau\right\}\Big]\tau d\tau$$
$$-\frac{1}{\pi(\nu v)^2}\xi_0^T E\phi - \frac{1}{\nu v}\left(2\xi_0^T D_s\phi + \frac{1}{\pi}\eta_0^T D_s\phi\right) + \frac{1}{\pi}\xi_0^T A\phi$$

$$g_{22}(\alpha,\nu v) = \frac{1}{\alpha\pi^2(\nu v)^2}\int_0^{2\pi}\Big[\theta^T \cos\tau\, g_0(\alpha,\nu v,\tau) - \left(\xi_0^T \sin\tau + \eta_0^T \cos\tau\right)\times$$
$$\times\left\{G_0^{(q)}(\alpha,\nu v,\tau)\theta\alpha\sin\tau + G_0^{(q')}(\alpha,\nu v,\tau)\theta\alpha\cos\tau\right\}\Big]\tau d\tau$$
$$+\frac{1}{\pi(\nu v)^2}\eta_0^T E\theta - \frac{1}{\nu v}\left\{2\eta_0^T D_s\theta - \frac{1}{\pi}\xi_0^T D_s\theta\right\} - \frac{1}{\pi}\eta_0^T A\theta$$

$$g_{30}(\alpha,\nu v) = \frac{1}{\alpha\pi(\nu v)^2}\phi^T\int_0^{2\pi} g_0(\alpha,\nu v,\tau)\cos\tau\, d\tau + \frac{1}{(\nu v)^2}\phi^T E\xi_0 - \frac{1}{\nu v}\phi^T D\eta_0 - \phi^T A\xi_0$$

$$g_{31}(\alpha,\nu v) = \frac{1}{\pi(\nu v)^2}\phi^T\left(\int_0^{2\pi}\left\{G_0^{(q)}(\alpha,\nu v,\tau)\cos^2\tau - G_0^{(q')}(\alpha,\nu v,\tau)\sin\tau\cos\tau\right\}d\tau\right)\phi$$
$$+\frac{1}{(\nu v)^2}\phi^T E\phi - \phi^T A\phi$$

$$g_{32}(\alpha,\nu v) = \frac{1}{\pi(\nu v)^2}\phi^T\left(\int_0^{2\pi}\left\{G_0^{(q)}(\alpha,\nu v,\tau)\sin\tau\cos\tau + G_0^{(q')}(\alpha,\nu v,\tau)\cos^2\tau\right\}d\tau\right)\theta + \frac{1}{\nu v}\phi^T D\theta$$

$$g_{40}(\alpha,\nu v) = \frac{1}{\alpha\pi(\nu v)^2}\theta^T\int_0^{2\pi} g_0(\alpha,\nu v,\tau)\sin\tau\, d\tau - \frac{1}{(\nu v)^2}\theta^T E\eta_0 - \frac{1}{\nu v}\theta^T D\xi_0 + \theta^T A\eta_0$$

$$g_{41}(\alpha,\nu v) = \frac{1}{\pi(\nu v)^2}\theta^T\left\{\int_0^{2\pi}\left\{G_0^{(q)}(\alpha,\nu v,\tau)\sin\tau\cos\tau - G_0^{(q')}(\alpha,\nu v,\tau)\sin^2\tau\right\}d\tau\right\}\phi - \frac{1}{\nu v}\theta^T D\phi$$

$$g_{42}(\alpha,\nu v) = \frac{1}{\pi(\nu v)^2}\theta^T\left\{\int_0^{2\pi}\left\{G_0^{(q)}(\alpha,\nu v,\tau)\sin^2\tau + G_0^{(q')}(\alpha,\nu v,\tau)\sin\tau\cos\tau\right\}d\tau\right\}\theta$$
$$+\frac{1}{(\nu v)^2}\theta^T E\theta - \theta^T A\theta \quad .$$

LIST OF SYMBOLS

A	inertia matrix
B	linear aerodynamic damping matrix
C	linear aerodynamic stiffness matrix
D	linear structural damping matrix
E	linear structural stiffness matrix
$F_0^{(q)}(\alpha,\nu,\tau)$	square matrix whose jth column is $(\partial f/\partial q_j)_{q=\alpha p_0}$
$F_0^{(q')}(\alpha,\nu,\tau)$	square matrix whose jth column is $(\partial f/\partial q_j')_{q=\alpha p_0}$
$G_0^{(q)}(\alpha,\nu v,\tau)$	square matrix whose jth column is $(\partial g/\partial q_j)_{q=\alpha p_0}$
$G_0^{(q')}(\alpha,\nu v,\tau)$	square matrix whose jth column is $(\partial g/\partial q_j')_{q=\alpha p_0}$
c_1, c_2	see equations (3-11) and (3-12)
c_{ij}	constant coefficients in equations (4-20) to (4-23) (no aerodynamic nonlinearity, *cf* Table 1)
$f(q,\nu q')$	column vector of nonlinear aerodynamic terms
$f_0(\alpha,\nu,\tau)$	see equation (4-7)
$g(q,\nu v q')$	column vector of nonlinear structural terms
$g_0(\alpha,\nu v,\tau)$	see equation (4-8)
$g_i(\alpha,\nu v)$	see equations (3-13) and (3-14)
$g_{ij}(\alpha,\nu v)$	coefficients in equations (4-20) to (4-23) (no aerodynamic nonlinearity, *cf* Table 1)
k_1, k_2	see equations (3-9) and (3-10)
k_{ij}	constant coefficients in equations (4-20) to (4-23) (no aerodynamic nonlinearity, *cf* Table 1)
$p(x,\tau)$	mode of trial solution for $q(\tau)$
$p_0(\tau) = p(0,\tau)$	
$q(\tau)$	column vector of generalised coordinates
$q_i(\tau)$	ith element of $q(\tau)$
t	time
v	ratio of airspeed to a reference speed
v_ℓ	flutter speed parameter of linear problem (*cf* equation (3-2))
x	see equation (2-10)
x_1, x_2,...	parameters in $p(x,\tau)$ (see also equations (4-2) to (4-4))
$y(q,q',q'')$	left hand side of equation (2-1)
z_1, z_2	coefficients in flutter mode of linear system (equation (4-1))
α	amplitude parameter
γ_{ij}	constant coefficients in equations (4-14) to (4-17) (no structural damping or nonlinearity)
ε	small parameter in equation (2-10)
η	column vector component of $p(x,\tau)$ (see equations (3-1) or (4-4))
$\eta_0 = (\eta)_{x_1=0}$	

LIST OF SYMBOLS (concluded)

θ	column vector in flutter mode of linear system (equation (4-1))
μ	coefficient in van der Pol equation
$\mu_{ij}(\alpha,\nu)$	coefficient in equations (4-14) to (4-17) (no structural damping or nonlinearity)
ν	frequency parameter
ν_{ℓ}	flutter frequency parameter of linear problem (*cf* equation (3-2)
ξ	column vector component of $p(x,\tau)$ (see equation (3-1) or (4-3))
$\xi_0 = (\xi)_{x_1=0}$	
$\tau = \omega t$	
ϕ	column vector in flutter mode of linear system (equation (4-1))
$\chi(\alpha,\nu,v,x)$	see equation (2-5)
ω	circular frequency of limit cycle

Dressings

'	indicates differentiation with respect to τ
a and s	subscripts indicate skew-symmetric and symmetric part of a square matrix respectively

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17. Abstract

A flutter analysis procedure for nonlinear systems is proposed as an alternative to timewise integration methods. It is based on an energy method due to J. Roorda and S. Nemat-Nasser; and shows promise of being a practical procedure provided the number of parameters can be minimised by the representation of the flutter system by an equivalent (condensed) two degree of freedom system in the neighbourhood of the critical condition. The relationship to the simpler method put forward by R.F. Taylor *et alia* is considered.

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